The Paneitz energy in dimension three revisited

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1 The Paneitz operator

On $\left(M^3, g\right)$, the Paneitz operator is given by

$$\begin{aligned} P\varphi &= \Delta^2\varphi + 4\operatorname{div}\left[Rc\left(\nabla\varphi, e_i\right)e_i\right] - \frac{5}{4}\operatorname{div}\left(R\nabla\varphi\right) \\ &\quad -\frac{1}{2}Q\varphi. \end{aligned}$$

Here e_1, e_2, e_3 is a local orthonormal frame with respect to g and

$$Q = -\frac{1}{4}\Delta R - 2\left|Rc\right|^2 + \frac{23}{32}R^2.$$

For any smooth positive function ρ,

$$P_{\rho^{-4}g}\varphi = \rho^7 P_g\left(\rho\varphi\right).$$

Hence if $\widetilde{g} = \rho^{-4}g$, then

$$\widetilde{Q} = -2\rho^7 P\rho.$$

It can be compared with conformal Laplacian operator in dimension $n \geq 3$,

$$L\varphi = -\frac{4(n-1)}{n-2}\Delta\varphi + R\varphi.$$

For $\rho > 0$,

$$L_{\rho^{\frac{4}{n-2}}g}\varphi = \rho^{-\frac{n+2}{n-2}} L_g(\rho\varphi).$$

If $\widetilde{g} = \rho^{\frac{4}{n-2}} g$, then

$$\widetilde{R} = \rho^{-\frac{n+2}{n-2}} L\rho.$$

2 Some examples

1. On $\mathbb{R}^3$, $P = \Delta^2$.

2. On S^3,

$$
\begin{aligned}
Q &= \frac{15}{8}, \\
P &= \Delta^2 + \frac{1}{2}\Delta - \frac{15}{16} = \left(-\Delta + \frac{3}{4}\right)\left(-\Delta - \frac{5}{4}\right).
\end{aligned}
$$

The eigenvalues

$$\lambda_1 = -\frac{15}{16} < 0, \quad \lambda_2 = \frac{105}{16} > 0.$$

Note on S^n, $n \geq 3$, $L > 0$. $S^n \backslash \{N\} \cong \mathbb{R}^n$. The H^1 capacity of $\{N\}$ is 0.

On S^3, $S^3 \backslash \{N\} \cong \mathbb{R}^3$. The H^2 capacity of $\{N\}$ is not 0, $H^2\left(S^3\right) \subset C^{1/2}\left(S^3\right)$.

3. On $S^2 \times S^1$,

$$Q = -\frac{9}{8},$$
$$P = \left(\Delta_{S^2} + \Delta_{S^1}\right)^2 + \frac{3}{2}\Delta_{S^2} - \frac{5}{2}\Delta_{S^1} + \frac{9}{16}.$$

We have $P > 0$ i.e. $\lambda_1 = \frac{9}{16} > 0$.

4. Berger spheres. $S^3 \cong SU(2)$.

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ X_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\ X_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

is a base for $\mathfrak{su}(2)$. For $t > 0$, we pick a left invariant metric with $t^{-1}X_1, X_2, X_3$ orthonormal. Then

$$R = 8 - 2t^2,$$
$$Q = -\frac{169}{8}t^4 + 41t^2 - 18.$$

For $0.8192 < t < 1.1269$,

$$Q > 0, \quad \lambda_1 < 0, \quad \lambda_2 > 0.$$

For $t > 1.1269$ or $0 < t < 0.8192$,

$$Q < 0, \quad \lambda_1 > 0.$$

3 The Q curvature equation and Paneitz energy

Let $\widetilde{g} = u^{-4} g$ for some $u > 0$, then $\widetilde{Q}$ =const becomes

$$Pu = \text{const} \cdot u^{-7}.$$

Let

$$\begin{aligned} & E(\varphi) \\ = & \int_M P\varphi \cdot \varphi d\mu \\ = & \int_M \left[(\Delta\varphi)^2 - 4Rc(\nabla\varphi, \nabla\varphi) + \frac{5}{4} R |\nabla\varphi|^2 - \frac{1}{2} Q\varphi^2 \right] d\mu, \end{aligned}$$

one possible way to solve the above equation is to study

$$Y_4(g) = \inf_{u \in H^2(M), u > 0} \left\| u^{-1} \right\|_{L^6(M)}^2 E(u).$$

This is simply the normalized total Q curvature functional (up to a negative constant).

If $Q > 0$ (for example S^3), it is not clear at all if $Y_4(g)$ is finite or not. The study of this question leads to the solution of Q curvature equation indirectly.

4 The standard sphere

Theorem 1 (Yang-Zhu) *For any* $u \in H^2\left(S^3\right)$ *with* $u > 0$*, we have*

$$\left\|u^{-1}\right\|_{L^6}^2 \int_{S^3}\left[(\Delta u)^2 - \frac{1}{2}|\nabla u|^2 - \frac{15}{16}u^2\right] d\mu \geq -\frac{15}{16}\left|S^3\right|^{\frac{4}{3}}.$$

In another word, the Paneitz energy minimizes at $u = 1$*.*

Note all critical points are identified by Choi-Xu.

Recall the sharp Sobolev inequality on S^n, $n \geq 3$,

$$\inf_{\varphi \in H^1(S^n)\setminus\{0\}} \frac{\int_{S^n}\left[\frac{4(n-1)}{n-2}|\nabla\varphi|^2 + n(n-1)\varphi^2\right] d\mu}{\|\varphi\|^2_{L^{\frac{2n}{n-2}}}}$$

$$= \quad n(n-1)|S^n|^{\frac{2}{n}}.$$

A classical approach is for $2 < q < \frac{2n}{n-2}$, we study

$$\inf_{\varphi \in H^1(S^n)\setminus\{0\}} \frac{\int_{S^n}\left[\frac{4(n-1)}{n-2}|\nabla\varphi|^2 + n(n-1)\varphi^2\right] d\mu}{\|\varphi\|^2_{L^q}}.$$

The perturbation problem can be easily shown to have a minimizer u. After scaling it satisfies

$$Lu = u^{q-1},\ u > 0 \quad \text{on } S^n.$$

Method of moving plane shows this equation has only constant solution. Hence

$$\begin{aligned} & \inf_{\varphi \in H^1(S^n) \backslash \{0\}} \frac{\int_{S^n} \left[\frac{4(n-1)}{n-2} |\nabla \varphi|^2 + n(n-1)\varphi^2 \right] d\mu}{\|\varphi\|_{L^q}^2} \\ = & \ n(n-1)|S^n|^{1-\frac{2}{q}}. \end{aligned}$$

Let $q \uparrow \frac{2n}{n-2}$, we get the sharp inequality.

5 Hint of the formulation of perturbation problem

Consider

$$Y_4(g) = \inf_{u \in H^2(M), u > 0} \left\| u^{-1} \right\|_{L^6(M)}^2 E(u).$$

Is $Y_4(g)$ achieved?

Let u_i be minimizing sequence. We assume $\max_M u_i = 1$. Then $E(u_i) \le c$ and $\|u_i\|_{H^2} \le c$. After passing to a subsequence $u_i \rightharpoonup u$ weakly in H^2, hence $u_i \rightrightarrows u$ uniformly.

- If $u > 0$, then it is a minimizer.

- If $u(p) = 0$ for some p, then

$$\begin{aligned} \max_M u &= 1, \\ \left\| u^{-1} \right\|_{L^6(M)} &= \infty, \\ E(u) &\le 0. \end{aligned}$$

Definition 2 *We say P satisfies condition P if for any $\varphi \in H^2(M) \setminus \{0\}$, $\varphi(p) = 0$ for some p, we have $E(\varphi) > 0$.*

Note

$$\text{condition P} \Longrightarrow Y_4(g) \text{ is achieved.}$$

- If $P > 0$, then condition P is satisfied. Hence $Y_4(g)$ is achieved (Xu-Yang).

- S^3 can not satisfy condition P because of the non-compact Mobius transformation group. But it satisfies condition NN:

  $\varphi \in H^2\left(S^3\right), \varphi(p) = 0$ for some $p \Rightarrow E(\varphi) \geq 0$.

  This should be compared with $\lambda_1 < 0$.

- $\mathbb{RP}^3$ satisfies condition P.

- Berger's sphere $\left(S^3, g_t\right)$ satisfies condition P except when $t = 1$ (Hang-Yang).

- Assume $Y(g) > 0$, $Q > 0$, then

$$(M, g) \text{ satisfies condition NN} \Longleftrightarrow \lambda_2 > 0;$$

and

$$\begin{aligned} & (M, g) \text{ satisfies condition P} \\ \Longleftrightarrow \; & \lambda_2 > 0 \text{ and } (M, g) \text{ is not conformal} \\ & \text{diffeomorphic to } S^3. \end{aligned}$$

(Hang-Yang)

6 The perturbation problem

For $\varepsilon > 0$ small, we replace P by $P + \varepsilon$,

$$E_\varepsilon(\varphi) = \int_{S^3} (P\varphi + \varepsilon\varphi)\, \varphi d\mu = E(\varphi) + \varepsilon \|\varphi\|_{L^2}^2 .$$

Then we study

$$-s_\varepsilon = \inf_{\varphi \in H^2(S^3), \varphi > 0} \left\| \varphi^{-1} \right\|_{L^6}^2 E_\varepsilon(\varphi) .$$

Clearly the extremal problem has a minimizer u with $\left\| u^{-1} \right\|_{L^6} = 1$ and

$$Pu + \varepsilon u = -s_\varepsilon u^{-7} \quad \text{on } S^3.$$

Conjecture 3 *For $\varepsilon > 0$ small, the above equation has only constant function as solution.*

Nevertheless we can show every minimizer must be constant function by symmetrization. Unlike classical symmetrization approach, our method only works for minimizers. Note

$$s_\varepsilon = \sup_{\varphi \in H^2(S^3), \varphi > 0} \left\| \varphi^{-1} \right\|_{L^6}^2 (-E_\varepsilon(\varphi)) .$$

Main ingredients:

- $$G_P(x,y) = -\frac{|x-y|}{8\pi}.$$

- For $\varepsilon > 0$ small enough,
$$G_{P+\varepsilon}(x,y) = -k_\varepsilon(x\cdot y),$$
here k_ε is a positive, strictly decreasing function on $[-1,1]$.

- (Baernstein-Taylor) If k is bounded decreasing function on $[-1,1]$, then for $f,g \in L^1\left(S^3\right)$,
$$\begin{aligned} &\int_{S^3}\int_{S^3} k(x\cdot y)\, f^*(x)\, g^*(y)\, d\mu(x)\, d\mu(y) \\ \leq\ &\int_{S^3}\int_{S^3} k(x\cdot y)\, f(x)\, g(y)\, d\mu(x)\, d\mu(y) \end{aligned}$$

- Let

$$Pv + \varepsilon v = -s_{\varepsilon} \left(u^{*} \right)^{-7},$$

then v is again an extremal function. In fact u must be radial symmetric and decreasing with respect to some point.

- Kazdan-Warner type conditions: if χ and ρ are positive smooth functions such that

$$P\rho = -\chi \rho^{-7},$$

then

$$\int_{S^3} \langle \nabla \chi (x), \nabla x_i \rangle \rho (x)^{-6} d\mu (x) = 0$$

for $i = 1, 2, 3, 4$.

This process uses ideas from Hang-Wang-Yan on maximizing isoperimetric ratios among conformal metrics with zero scalar curvature and Robert on the positivity of minimizers for fourth order Q curvature problem in dimension at least 5.

7 General metrics

Theorem 4 (Hang-Yang) *If $\left(M^3, g\right)$ is a smooth compact manifold with $Y(g) > 0$ and there exists a $\widetilde{g} \in [g]$ such that $\widetilde{Q} > 0$, then the following statements are equivalent*

- $Y_4(g) > -\infty$.
- $\lambda_2(P) > 0$.
- P *satisfies condition NN.*

The first ingredient is an identity by Hang-Yang: if $Y(g) > 0$, then

$$P\left(G_{L,p}^{-1}\right) = -256\pi^2 \delta_p + G_{L,p}^{-1} \left| Rc_{G_{L,p}^4 g} \right|_g^2 .$$

Here are some applications of the identity:

1. Assume $Y(g) > 0$ and there exists a $\widetilde{g} \in [g]$ such that $\widetilde{Q} > 0$, then $\ker P = 0$ and

$$G_P = H + \sum_{k=1}^{\infty} \Gamma_k * H,$$

here

$$\begin{aligned}
\Gamma_k &= \Gamma_1 * \cdots * \Gamma_1 \; (k \text{ times}), \\
H(p,q) &= -\frac{G_L(p,q)^{-1}}{256\pi^2}, \\
\Gamma_1(p,q) &= \frac{G_L(p,q)^{-1}}{256\pi^2} \left| Rc_{G_{L,p}^4 g} \right|_g^2 (q).
\end{aligned}$$

Also

$$K_1 * K_2(x,y) = \int_M K_1(x,z) K_2(z,y) \, d\mu(z).$$

Here we use ideas of Aubin. The formulas are motivated from Humbert-Raulot and Gursky-Malchiodi.

2. Assume $Y(g) > 0$. Denote

$$T_{\Gamma_1} f(p) = \int_M \Gamma_1(p,q) f(q) \, d\mu(q).$$

The following statements are equivalent:

- there exists a $\widetilde{g} \in [g]$ such that $\widetilde{Q} > 0$;
- the spectral radius $r_\sigma \left(T_{\Gamma_1} \right) < 1$;
- $\ker P = 0$ and $G_P(p,q) < 0$ for $p \neq q$.

The second ingredient is:

$$\nu(g) = \inf_{\substack{u \in H^2(M) \setminus \{0\} \\ u(p)=0 \text{ for some } p}} \frac{E(u)}{\int_M u^2 d\mu}.$$

Example 5

$$\begin{aligned} P \textit{ satisfies condition NN} &\Leftrightarrow \nu(g) \geq 0; \\ P \textit{ satisfies condition P} &\Leftrightarrow \nu(g) > 0. \end{aligned}$$

Example 6 $\nu\left(S^3, g_{S^3}\right) = 0$.

Shectch of proof. Note that

$$\lambda_1 \leq \nu \leq \lambda_2.$$

If $\lambda_2 > 0$, let $u \in H^2(M)$ such that $u(p) = 0$ for some p, $\|u\|_{L^2} = 1$ and $E(u) = \nu$. Then based on the sign of G_P we know $u(q) \neq 0$ for $q \neq p$, say $u(q) > 0$ for $q \neq p$. We also have

$$Pu = \nu u + \alpha \delta_p.$$

Hence

$$\int_M Pu \cdot G_{L,p}^{-1} d\mu = \nu \int_M u G_{L,p}^{-1} d\mu.$$

On the other hand,

$$\int_M Pu \cdot G_{L,p}^{-1} d\mu = \int_M u G_{L,p}^{-1} \left| Rc_{G_{L,p}^4 g} \right|_g^2 d\mu.$$

It follows that

$$\nu \int_M u G_{L,p}^{-1} d\mu = \int_M u G_{L,p}^{-1} \left| Rc_{G_{L,p}^4 g} \right|_g^2 d\mu.$$

Hence $\nu \geq 0$. ■